On the fundamental nature of the state parameter

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The state parameter ψ is widely used for soil characterisation and as a controlling parameter in modern constitutive understanding of soil, but there remains a perception that the control of soil strength by ψ is merely that of a correlation. This perception possibly stems from ψ having been introduced from 'principles' of critical state theory rather than derived, which is now rectified. It is shown that the control of limiting dilatancy by the state parameter (and thus soil strength through stress–dilatancy) is a formal mathematical consequence of Casagrande's canonical characterisation linking void ratio to soil constitutive behaviour. This formal consequence is independent of soil type, being applicable across the spectrum from clays to sands. Three dimensionless and familiar soil properties are involved in addition to those characterising the critical state locus: M_{tc}, N and X. The framework is kinematic, with no constitutive model: it is a constraint on models. Example data are shown for sands, silts and clays to illustrate the independence of the theory from geological descriptors.

KEYWORDS: clays; sands; silts; soil mechanics & constitutive models

INTRODUCTION

The state parameter ψ is widely used for soil characterisation and as a controlling parameter in modern constitutive understanding of soil, but there remains a perception that the control of soil strength by ψ is merely that of a correlation. This perception possibly stems from ψ having been introduced from 'principles' of critical state soil theory (CST). Here a theoretical derivation is given to clarify the fundamental nature of ψ, also emphasising that CST, at its simplest, links soil strength to void ratio and can be understood without resort to constitutive models, although stress–dilatancy is essential for the full perspective. In what follows all stresses are 'effective'.

SOIL STRENGTH

Soil comprises particles, which are easily seen in the case of sands but require a microscope for finer-grained soils. Idealising soils as a collection of equal-sized spheres leads to the kinematic result that soils must generally change volume as they deform, as illustrated on Fig. 1.

Soil often displays an apparent cohesion: the c', ϕ' strength idealisation. The 'cohesion' is obviously fictional, however, because there are no bonds between soil particles. Taylor (1948) used the kinematic idea of Fig. 1 to suggest that soil strength comprised two components: 'friction' and 'interlocking'; Bishop (1950) formalised Taylor's idea, Fig. 2.

The notion that work is only dissipated by the distortional strain rate is implicit in Taylor–Bishop, but is not quite correct; originally stated in the context of a shear-box test, the Taylor–Bishop strength model generalises as

$$\eta_{max} = M - (1 - N)\, D_{min} \tag{1}$$

where N was introduced by Nova (1982) and represents the work increment not dissipated by distortional strain (Jefferies, 1997); Taylor–Bishop is recovered if $N = 0$. The 'maximum dilatancy' seen in Fig. 2 is actually the limiting strain-rate ratio D_{min} in equation (1) for consistency with the compression-positive convention of soil mechanics; η is the ratio of deviatoric to mean stress with $\eta = M$ corresponding to the critical state. Fig. 3 shows some examples of sand behaviour and the fit of equation (1) to the data; the behaviour of all sands is captured by equation (1) and the data shown are not exceptional.

The issue then is to what extent Taylor–Bishop applies to fine-grained soils. The 1950s saw a sustained and fundamental investigation of clay behaviour, directed by Henkel (see his obituary in *Géotechnique* (Anon, 2006)), using remoulded samples and with extensive use of drained tests which provide dilatancy data; Parry (1956) is a convenient summary. These data for Weald Clay and London Clay are also shown in Fig. 3, and an equal fit of equation (1) is found for these clays as for sands.

Some recent data on silt strength have also been added to Fig. 3 to illustrate the independence of equation (1) from soil type. Equation (1) is a simple physical idealisation for dissipation of 'frictional' work, but this cannot be treated as fundamental without knowing why a particular D_{min} develops. Experience is that denser soils dilate more, but experience is not fundamental understanding. The link is the critical state, which is implicit in Taylor–Bishop (see Fig. 2).

CRITICAL VOID RATIO

Hydraulic fill dams were a common construction method in North America during the late 1800s to early 1900s, despite their propensity to suddenly fail by liquefaction slides during construction (e.g. Hazen (1920), and subsequent discussion). A contractive response to loading will cause positive excess pore pressure in saturated soil, so Reynold's idealisation (Fig. 1) underlay engineering of the first liquefaction-resistant dam at Franklin Falls by the US Army Corp of Engineers (USACE; Lyman, 1938) with a focus on testing to determine the safe (= 'critical') density for the constructed works. Casagrande's canonical summary of the Franklin Falls testing is shown in Fig. 4 as the evolution of void ratio and stress in a shear box test; a common end point, for all initial conditions, was identified as the critical

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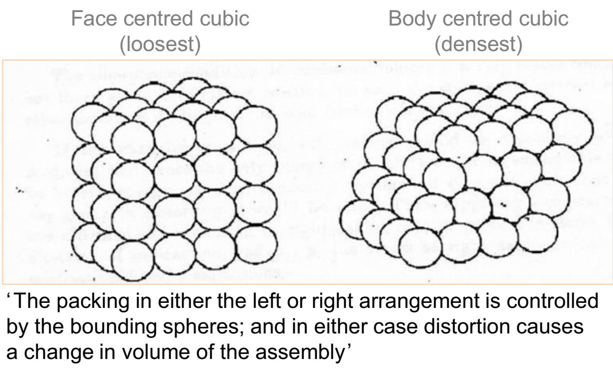

Fig. 1. Reynolds (1885) kinematic explanation of dilation

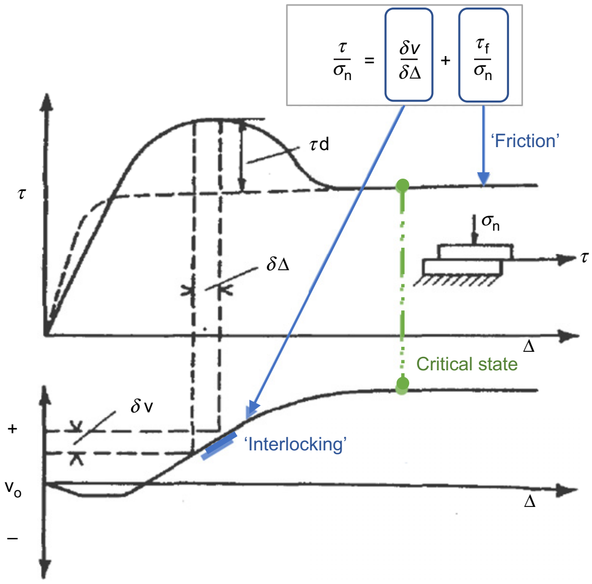

Fig. 2. Soil strength related to rate of volume change in direct shear (after Bishop (1950); annotated by the present author)

void ratio (e_c) and comprises two conditions, which generalise into three dimensions as

$$D = \frac{d\varepsilon_v}{d\varepsilon_q} = 0 \ldots \text{deformation at constant void ratio} \quad (2a)$$

$$\frac{dD}{d\varepsilon_q} = 0 \ldots \text{which can continue without limit} \quad (2b)$$

where ε_q is the deviatoric strain suggested by Resende & Martin (1985).

A liquefaction slide developed at Fort Peck dam in 1938, near concurrent with Franklin Falls, and the USACE's investigation of the fill involved found that the critical void ratio e_c depended on the effective confining stress (Middlebrooks, 1942) – a dependence today known as the critical state locus (CSL), with e_c reasonably represented by

$$e_c = \Gamma - \lambda \ln(p) \quad (3)$$

where Γ, λ are soil properties. The 1950s investigation of clay behaviour referred to above also provided the critical state of London and Weald clays (Parry, 1958), with equation (3) being a good representation of their CSLs. The 1980s saw large-scale use of hydraulic sand fills in the Canadian Arctic and the engineering of those fills directly followed the pioneering work of the USACE (Jefferies *et al.*, 1988). Considerable data were obtained on the CSL of sands.

Figure 5 presents a selection of CSLs for various soils, from sands to clays. Two points are clear: (*a*) representing a CSL using equation (3) is a reasonable approximation for at least a one order of magnitude range in *p*; and (*b*) the adequacy of equation (3) is independent of soil type.

DILATANCY AND THE CRITICAL STATE

It seems usual for 'stress' to be the starting point for understanding soil behaviour, possibly from the days of Coulomb when this subject started from consideration of soil strength in the stability of earthworks; this can be seen

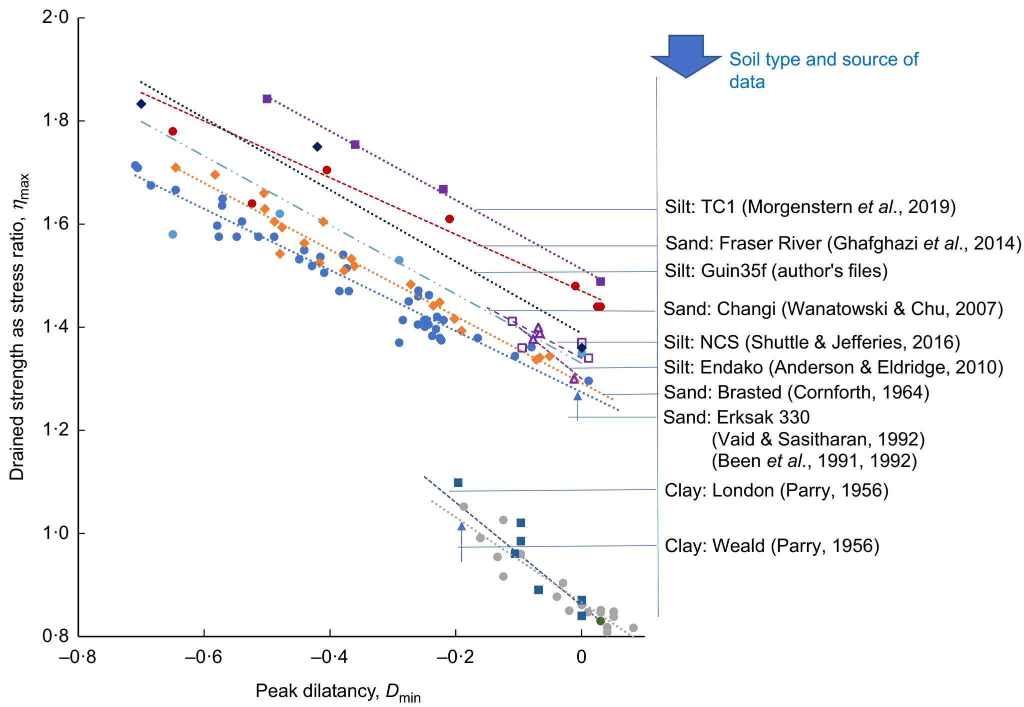

Fig. 3. The effect of dilatancy on the drained strength of several soils. Trend lines are equation (1) using soil properties given in Table 1

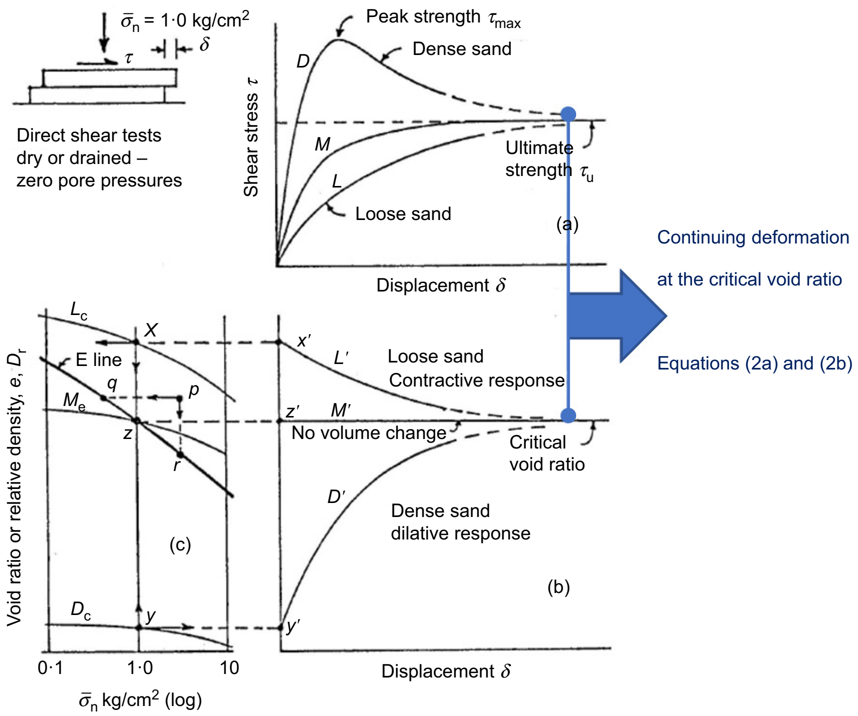

Fig. 4. Origin and meaning of the critical void ratio (Casagrande, 1975) (as updated from Casagrande (1936); annotated by the present author)

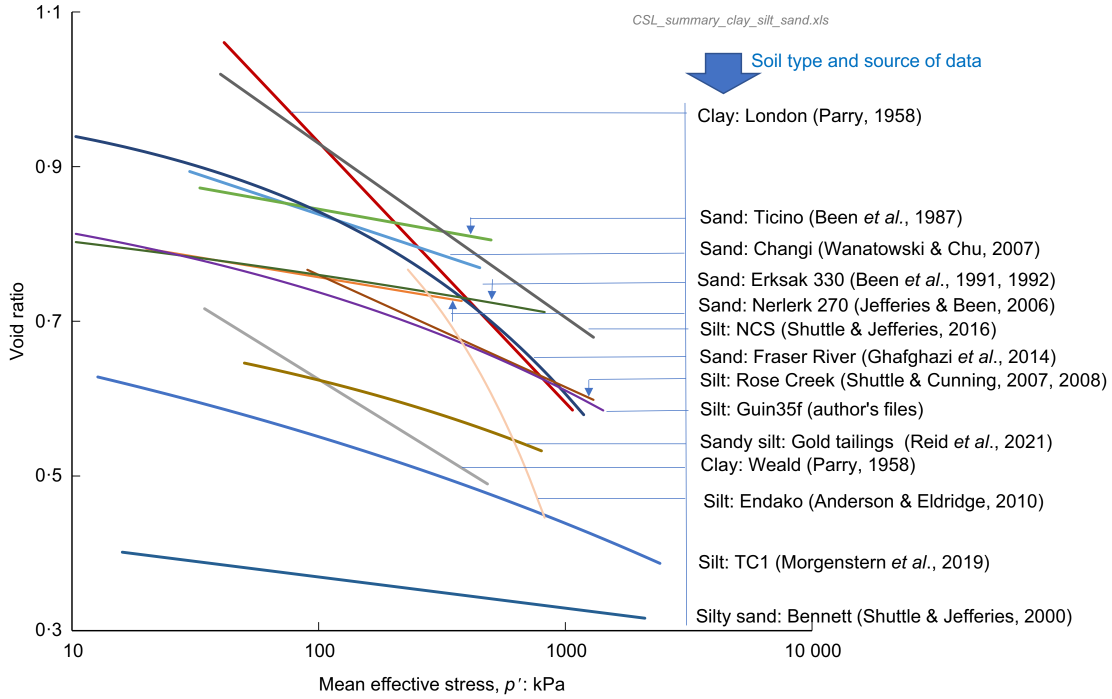

Fig. 5. Critical state loci of sands, silts and clays (each locus shown only over the range it was measured)

reflected in Casagrande's labelling of his figure as a sequence 'a' to 'c' where 'stress' (Fig. 4(a)) is followed by 'void ratio change' (Fig. 4(b)), which leads to the 'critical state' (Fig. 4(c)). But, Reynold's insight (Fig. 1) is that soil behaviour is controlled by how particles can move: kinematics; the evolution of stress (and thus strength) is then the result, not the start. This idea of alternative ways of looking at soil behaviour is actually common although unappreciated: triaxial tests can be carried out using load control or displacement control with the data remaining equally valid whichever choice is made. Likewise, mathematically representing the physics of soil behaviour can also be approached in either way.

A kinematic approach naturally leads to the idea that Casagrande's canonical figure (Fig. 4) should be read the opposite way: that is, the reverse sequence 'c' to 'a'. This alternative view is crystallised by casting the framework as two axioms for subsequent mathematical development. (*a*) Axiom 1: a unique CSL exists for any soil. (*b*) Axiom 2: soil evolves to its CSL with distortional strain.

Axiom 1 is just Fig. 4(c) that gives the CSL, which forms the end point of all distortional paths. The power of this reference state is that a soil's CSL is unique, where 'unique' merely requires e_c being a single-valued function of the principal stresses. In terms of physics, if the end state is known (defined) then the mathematics becomes well conditioned (i.e. easy and stable). Physical reasonableness (loose soil should be weaker than dense soil) requires the CSL to be monotonic decreasing with *p*.

Axiom 2 reflects Fig. 4(b) and only involves two variables, *e* and ε_q. Defining a 'state parameter' ψ as the void ratio offset from the soil's end state (critical state) at the current mean effective stress (e_c)

$$\psi = e - e_c \tag{4}$$

Axiom 2 explicitly and simply represents Fig. 4(b), for all soil-specific CSL, as

$$\psi \rightarrow 0 \text{ as } \varepsilon_q \rightarrow \infty \tag{5}$$

Figure 4(c), which defines e_c, is a 'state diagram' showing where a soil exists in its accessible space; strictly there should be lines denoting the lowest void ratio and the greatest void ratio to define accessible space, both of which may depend on mean stress, but these are commonly omitted. Importantly, however, in a state diagram there is no concept of yielding (plasticity), even though strains may actually be largely plastic; thus, one must start with total deviatoric strain in equation (5). The important separation into elastic and plastic components will be addressed later.

Equation (5) gives a rule for void ratio evolution. The rate of evolution of void ratio is dilatancy, with deviatoric stress (and thus η) arising through stress–dilatancy giving Fig. 4(a); this aspect is presented later after developing the role of ψ. The starting point is to formalise the trend of equation (5) as a differential equation

$$\mathrm{d}\psi/\mathrm{d}\varepsilon_q = -f(\psi) : f(0) = 0 \text{ and } \mathrm{d}f/\mathrm{d}\psi > 0 \tag{6}$$

The simplest function complying with the conditions in equation (6) is

$$f(\psi) = \chi\,\psi \tag{7}$$

where χ is a material constant (i.e. soil property). Higher odd-order terms could be added to equation (7), giving a series expansion of all generally admissible *f*(), but Occam's razor is that this should only be done if the simplest form proves inadequate with respect to measurements – which turns out to not be the case, as will be shown. Combining equations (6) and (7) then taking the differential of equation (4) and substituting

$$\mathrm{d}e - \mathrm{d}e_c = -\chi\,\psi\,\mathrm{d}\varepsilon_q \tag{8}$$

On dividing equation (8) through by the current specific volume and rearranging

$$\frac{\mathrm{d}e}{1+e} = \frac{\mathrm{d}e_c}{1+e} - \frac{\chi\psi\mathrm{d}\varepsilon_q}{1+e} \tag{9}$$

The left-hand term of equation (9) is the volumetric strain increment, so on dividing through by the deviatoric strain (allowing for the compression positive convention)

$$D = \frac{\mathrm{d}\varepsilon_v}{\mathrm{d}\varepsilon_q} = \frac{\chi\psi}{1+e} - \frac{1}{1+e}\frac{\mathrm{d}e_c}{\mathrm{d}\varepsilon_q} \tag{10}$$

It is possible to proceed using any admissible form of CSL, but a simple result is obtained by adopting equation (3) (which is at least a good approximation). Differentiating equation (3)

$$\mathrm{d}e_c = -\lambda\frac{\mathrm{d}p}{p} \tag{11}$$

Substituting equation (11) in equation (10)

$$D = \frac{\chi\,\psi}{1+e} + \frac{\lambda}{p(1+e)}\frac{\mathrm{d}p}{\mathrm{d}\varepsilon_q} \tag{12}$$

Equation (12) is general, but cannot be used without knowing how ψ and *p* evolve. For the limited purpose of understanding the drained peak strength of soil, equation (1), however, it is sufficient to limit attention to conditions at peak strength – in a *drained* triaxial test with continuously increasing distortional strain, at peak strength the condition $\mathrm{d}p/\mathrm{d}\varepsilon_q = 0$ develops as the stress path reverses direction. From equation (1) peak strength corresponds to 'maximum' dilation, which is D_{min} because of the compression positive convention. Because the stress state is stationary at q_{max} there are no elastic strain increments. Thus, axiom 2 has the corollary

$$D_{min} = D^{p}_{min} = \frac{\chi\,\psi}{1+e} : \text{ the limiting dilatancy in drained compression} \tag{13}$$

Historically, the '$1+e$' term was omitted from the definition of the rate coefficient in equation (13), which is unfortunate in principle. Practically, however, it seems sufficient to regard '$1+e$' as 'bundled in' to the coefficient χ; thus, the simplification consistent with current use

$$D_{min} = \chi\,\psi_{D_{min}} \tag{14}$$

The subscript on ψ in equation (14) emphasises that ψ must be taken as concurrent with D_{min} (i.e. it is not ψ_0, the value at the start of the strain path). Equation (14) fits every soil type, some example data being shown in Fig. 6; properties of these soils are given in Table 1.

STATE CONTROL OF SOIL BEHAVIOUR

The state parameter is kinematic in principle, controlling dilatancy not strength; strength comes through substituting equation (14) in equation (1) giving

$$\eta_{max} = M(\theta)\left[1 - (1-N)X\,\psi_{D_{min}}/M_{tc}\right] \tag{15}$$

where η_{max} is the limiting strength of the soil (the Hvorslev surface); θ is the Lode angle characterising the proportion of intermediate principal stress; and M_{tc}, *N*, *X* are the soil's properties as measured in triaxial compression.

There are differing views on $M(\theta)$ – for example, Matsuoka & Nakai (1974) as opposed to simply constant ϕ_c – but there is some operational similarity across the various views, with Jefferies & Shuttle (2011) suggesting a sufficient representation is

$$M(\theta) = M_{tc}\left[1 - \frac{\cos(3\theta/2 + \pi/4)}{1 + 3/M_{tc}}\right] \tag{16}$$

Implicit in equation (15) is the relation

$$D(\theta)_{min} = \frac{M(\theta)}{M_{tc}}D(\pi/6)_{min} \tag{17}$$

which was developed by Jefferies & Shuttle (2002) from the theoretical requirement of 'closed and convex' yield surfaces (a yield surface cannot have gaps or fold back on itself). In this, triaxial compression ($\theta = \pi/6$) has been taken as the

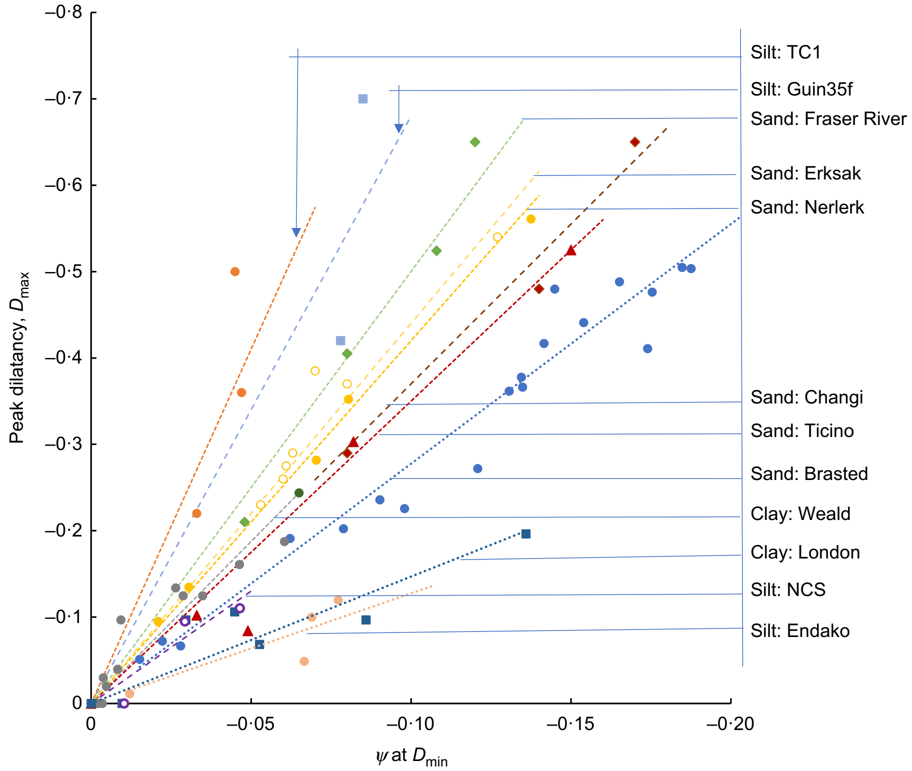

Fig. 6. State–dilatancy of soils in drained triaxial compression. Trend lines are equation (14) using soil properties given in Table 1

reference condition, consistent with how M_{tc} is defined as the soil property and with the corresponding requirement that X is also determined under triaxial compression.

Equation (15) corresponds to the kernel strength trend of Been & Jefferies (1985, 1986), but then leaves open the wider question of the role of ψ in the situation $\eta \neq \eta_{max}$; more generally, as equation (14) is a state–dilatancy rule (hardening law) how does equation (14) relate to soil behaviour that is commonly understood in terms of stress–dilatancy? What is the role of η?

Although it might be thought that true understanding of soil behaviour can only come from knowledge of how forces are transmitted through the chain of contacts between the soil particles (a micromechanics view), there is something more fundamental: the second law of thermodynamics. This law has its origins in Victorian engineers developing better steam engines, but has now progressed to being included in modern physics as a principle (entropy must always increase) almost of the stature of relativity and now influencing the understanding of the universe at cosmological scale (e.g. Davies, 1977). In engineering mechanics, plastic strains dissipate work into heat, which is then lost to the system (repeatedly bend a small piece of metal with your hands and feel the ductile zone of plastic deformation warm up); thus, the dissipation function is the key to understanding. Schofield & Wroth (1968) introduced these ideas to soil mechanics, assuming co-axiality of stress and strain increments (always true in the triaxial test, but possibly not so in general loading), and their approach can be expressed (Jefferies & Been, 2006) as

$$\Omega = \frac{\mathrm{d}W^{p}}{p\ \mathrm{d}\varepsilon_{q}^{p}} = \eta + D^{p} \qquad (18)$$

where Ω is the normalised (dimensionless) rate of dissipation of plastic work ($\mathrm{d}W^{p}$) into heat. This reliance on thermodynamics is why the work-conjugate Resende & Martin (1985) strain invariants define D used in equation (18); in reality, the Resende & Martin strain invariants are familiar, as they become the ‘Cambridge’ invariants of Schofield & Wroth (1968) under the symmetry of triaxial conditions. The Resende & Martin invariants are also linear, thus allowing direct use of the elastic–plastic strain decomposition: $\mathrm{d}\varepsilon_{q} = \mathrm{d}\varepsilon_{q}^{e} + \mathrm{d}\varepsilon_{q}^{p}$ and $\mathrm{d}\varepsilon_{v} = \mathrm{d}\varepsilon_{v}^{e} + \mathrm{d}\varepsilon_{v}^{p}$.

Equation (18) is a simple algebraic transform of the rate of plastic working on an element of soil and involves no constitutive model, making this basic form of stress–dilatancy a consequence of thermodynamics: Ω is the thermodynamic control parameter. The issue then becomes: what is an appropriate representation of Ω? Schofield & Wroth (1968) suggested the idealisation

$$\Omega = M_{tc} \quad \text{. . . the work dissipation postulate of original Cam Clay} \qquad (19)$$

Substitution of equation (19) into equation (18) gives a simple stress–dilatancy relation, which generalises the Taylor–Bishop peak-strength idealisation to a continuous stress–dilatancy rule, as illustrated on Fig. 7.

Equation (14) establishes the limiting dilatancy, but does so in terms of the state parameter at that limit – which in general will not be known at the outset for boundary value problems (although always known in the results of triaxial tests). However, it is sufficient and convenient to regard equation (14) as an evolving limit as it is not necessary to know that limit on the way to reaching it – use equation (14) with the current state parameter, which will automatically become $\psi_{D_{min}}$ as void ratio changes with strain, as illustrated in Fig. 7. When the evolution of dilation reaches its limit (peak strength), stress–dilatancy continues to operate but with the evolution of D now controlled by equation (14) and which itself depends on how ψ evolves with volumetric strain and stress path: Casagrande’s strength evolution with void ratio of Fig. 4(a).

It has been known since at least Rowe (1962) that the coefficient in stress–dilatancy (here $M_{i} \leftrightarrow \phi_{f}$ in Rowe’s

Table 1. Properties of soils whose behaviour is shown on Figs 3 and 6

Soil	Geological					Mechanics			
	D_{50}: μm	<60 μm	<2 μm	C_U	PI*	λ_{10}†	M_{tc}	N	X
Sand: Brasted	240	<2%	—	2·3	—	0·050	1·29	0·35	2·8
Sand: Changi	320	0·4%	—	2·0	—	0·106	1·33	0·33	3·7
Sand: Erksak	330	0·7%	—	1·8	—	0·040	1·26	0·37	4·4
Sand: Fraser River	300	~1%	—	1·7	—	0·090	1·47	0·45	5·0
Sand: Nerlerk	270	1·7%	—	1·9	—	0·053	1·26	0·38	4·2
Sand: Ticino	530	0%	—	1·8	—	0·057	1·27	0·35	3·5
Silt: Endako	8	99%	0%	18	23%	0·541	1·35	0·35	0·8
Silt: Guin35f	115	34%	0%	6	—	0·102	1·39	0·30	6·8
Silt: TC1	60	55%	15%	90	5%	0·105	1·52	0·35	8·2
Silt: NCS	10	97%	10%	6	—	0·225	1·34	0·35	2·6
Clay: London	—	100%	50%	—	52%	0·340	0·86	0·00	1·5
Clay: Weald	—	100%	40%	—	25%	0·198	0·86	0·15	2·5

*Plasticity index (PI), which has nothing to do with the Theory of Plasticity
†The property λ_{10} (= 2·3λ) is the best fit over the stress range 40–400 kPa and is only given as an index of soil compressibility.

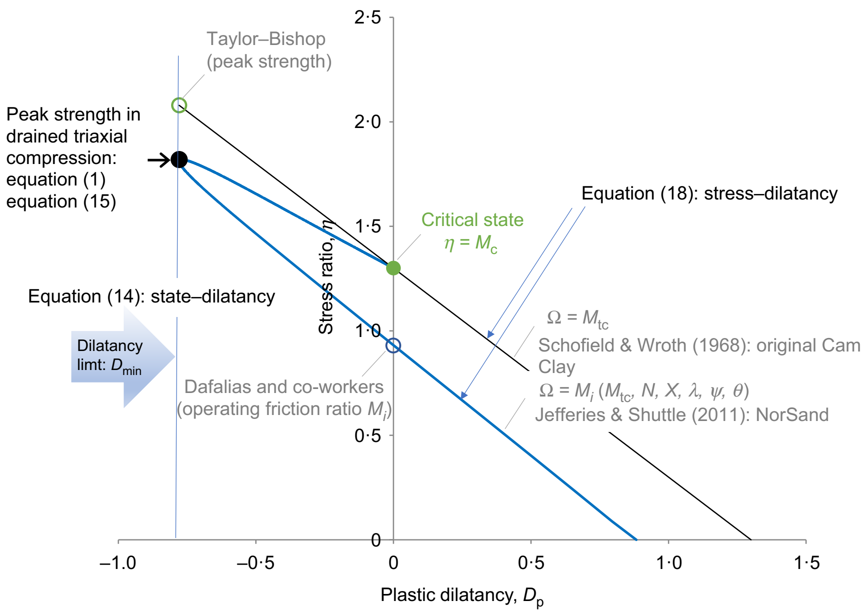

Fig. 7. Behaviour of dense soil ($\psi_0 = -0{\cdot}2$, $M_{tc} = 1{\cdot}3$) in drained triaxial compression and also illustrating key historical contributions

notation) evolves with strain, from a lower limit corresponding to the mineral–mineral sliding friction of the soil particles to the critical state dissipation of plastic work as distortional strain develops

$$M_i \rightarrow M(\theta) \text{ as } \varepsilon_q \rightarrow \infty \tag{20}$$

This 'operating' zero-dilatancy coefficient is denoted as M_i with the subscript i indicating an 'image' of the critical state as M_i honours equation (2a) while violating equation (2b); this condition is also called the 'phase change' or 'quasi steady-state' in the literature, both of which are misleading names for the physics involved.

Rules/models for the detail of equation (20) have proved elusive. And, indeed this is where one might expect soil 'fabric' to show its influence. A key contribution about how M_i evolves was provided by Dafalias and co-workers (Manzari & Dafalias, 1997; Li *et al.*, 1999), who suggested that equation (20) be replaced by

$$M_i(\psi) \rightarrow M(\theta) \text{ as } \psi \rightarrow 0 \tag{21}$$

where the parameter M_d of bounding surface theory has been taken as the same thing as M_i of CST; and note that $M(\theta)$ does not depend on ψ, see equation (16), which constrains idealisation of M_i. Jefferies & Shuttle (2011) expanded on these ideas by viewing equation (21) as a modification of Schofield & Wroth (1968) that gives

$$\Omega = M_i(M_{tc}, N, X, \lambda, \psi, \theta) \tag{22}$$

The Appendix gives details of the current M_i (and modelling choices). An example of stress–dilatancy resulting from the using equation (22) in equation (18) is illustrated in Fig. 7; the same control of equation (14) arises and with a now weak dependence of work dissipation on ψ. This computed trend varies with soil state and properties, with that shown in Fig. 7 being computed for $\psi_0 = -0{\cdot}2$ using the properties $M_{tc} = 1{\cdot}3$, $N = 0{\cdot}35$, $X = 5{\cdot}0$ and $\lambda = 0{\cdot}02$.

DISCUSSION

A particular feature of equation (14) is that it applies before localisation develops, with D_{min} commonly arising before 5% axial strain. This provides the interesting result that the applicability of a CSL determined from tests on loose samples to dense soil behaviour can be checked by observing whether the trend in $D_{\min}$, ψ pairs from tests on dense samples goes through the origin; see Fig. 6.

Equation (15) links contributions from Reynolds (1885), Lyman (1938), Taylor (1948), Bishop (1950) and Casagrande (1975) into a single framework of how void ratio controls soil strength under arbitrary conditions. Easily understood properties are involved, all measured using a few triaxial compression tests. The framework is purely kinematic and applies to all soils provided that there are no bonds between particles (residual soils may behave a little differently from the current CST framework); derived through mechanics, there is no role for geologic descriptors (e.g. 'fines content') with different soils simply represented through the numerical values of their properties.

Representing material behaviour, including soil, by plasticity always involves three aspects: yielding (when do plastic strains arise?), hardening (how does the yield criterion evolve with strain?) and a flow rule (giving the relative proportion of the plastic strain increments). Within any consistent model for soil, state–dilatancy equation (14) is a constraint on hardening, whereas stress–dilatancy equation (18) is a constraint on the flow rule. Within this consistent framework, differing idealisations (models) of soil behaviour revolve around how these constraints are honoured.

There seem to be only two choices for using equation (14) and these are to implement it through equation (15) as a limiting strength (giving 'bounding surface' models) or as a direct constraint on yield surface hardening by way of normality (giving 'generalised Cambridge' models). There is considerable similarity between these approaches.

There are rather more choices regarding equation (18), all depending on the idealisation of Ω. For example, void ratio is generally not viewed as influencing M_{tc}, even though data on some soils suggest it does so (e.g. Been *et al*., 1991, 1992). Further, there is abundant experimental evidence that the details of the arrangement of soil particle contacts ('fabric') affects M_i when $\psi \neq 0$, with the central problem not being theoretical understanding, but rather the absence of a test method to characterise these contact arrangements. Improvements to the form of Ω. should be anticipated, and micromechanical insights would be helpful in achieving this.

The derivations above have assumed constant principal stress direction and co-axiality of stress and strain increment; these conditions are met in triaxial and biaxial laboratory tests, but not in general. Principal stress rotation (PSR) affects soil behaviour, and arises in most practical engineering, but the fundamental insights of Arthur and co-workers (Arthur *et al*., 1980) about PSR are widely neglected; Jefferies *et al*. (2015) added the further axiom that PSR always softens soil behaviour, but that remains a rare contribution to this aspect of soil behaviour. Non-coaxiality also arises pervasively in practice as well, as in the presently popular direct simple shear test in the laboratory; non-coaxiality is easier in principle, requiring a vector mapping of strain rates in the deviatoric strain invariant ε_{q} (Guiterrez & Ishihara, 2000) with the work dissipation equation (18) then continuing as given. It is also useful to note that, being anchored to void ratio, CST is intrinsically large strain. Thus, the framework set out here, although derived in the context of the triaxial tests, sets out necessary constraints when representing the effect of any general loading on soil.

The symbol ψ is widely used to represent maximum or characteristic dilation angle, particularly in the non-associated Mohr–Coulomb (NAMC) model of soil behaviour; for clarity call this angle ψ_{NA}. Equally, ψ is often used as the symbol for 'potential' in physics, and this association with potential was the basis of choosing ψ to denote the state parameter in the paper by Been & Jefferies (1985, 1986) since this notion is in state space, for clarity this potential should be called ψ_{e}. The dilation angle is associated with strain rates at peak strength and, in the case of triaxial compression, is (Schanz & Vermeer, 1996)

$$\begin{aligned} \sin\psi_{\mathrm{NA}} &= -\frac{\mathrm{d}\varepsilon_{\mathrm{v}}/\mathrm{d}\varepsilon_1}{2-\mathrm{d}\varepsilon_{\mathrm{v}}/\mathrm{d}\varepsilon_1} \ @\ \eta_{\max} \\ &= -D_{\min}/(2-D_{\min}/3) \end{aligned} \tag{23}$$

On substituting equation (14) in equation (23), the equivalence of the conventionally understood dilation angle (in triaxial compression) and the state parameter are found

$$\sin\psi_{\mathrm{NA}} = -\chi\psi_{\mathrm{e}}/(2-\chi\psi_{\mathrm{e}}/3) \tag{24}$$

It is both pleasing, and a useful reminder, to see potential and dilation angle formally linked; Fig. 8 illustrates this linking.

HISTORICAL NOTE

Parry (1958: p. 185) stated 'the volume change at failure under drained test conditions and the pore-pressure change at failure under undrained test conditions are controlled, both in rate and sense, by the relation between the mean effective stress and the water content for any clay'. Been & Jefferies (1985, 1986) applied this notion by taking 'the relation' as the initial void ratio offset from the CSL, ψ_0 in the current notation, as opposed to the stress ratio p_c/p used by Parry. While convenient for presenting the results of laboratory tests, ψ_0 is not quite correct: ψ at $D_{\min}$ should be used as derived above (a little data processing is needed). Fig. 9 shows the data of Been & Jefferies (1985, 1986) in this correct form with the trend line computed using 'not unusual' values $M_{\mathrm{tc}} = 1{\cdot}3$, $N = 0{\cdot}35$, $X = 4{\cdot}0$ (which of course vary somewhat from soil to soil, which is why careful inspection of the figure will show some soils consistently plotting above the trend drawn while others lie below).

Been & Jefferies (1985, 1986) further asserted that soil frictional strength was independent of soil type when ϕ was expressed in terms of ψ. However, they were careful to note the possible consequence of a missing fabric parameter (Been & Jefferies, 1985, 1986) and gave a band for soil behaviour rather than a best-fit trend line; the effect of fabric was

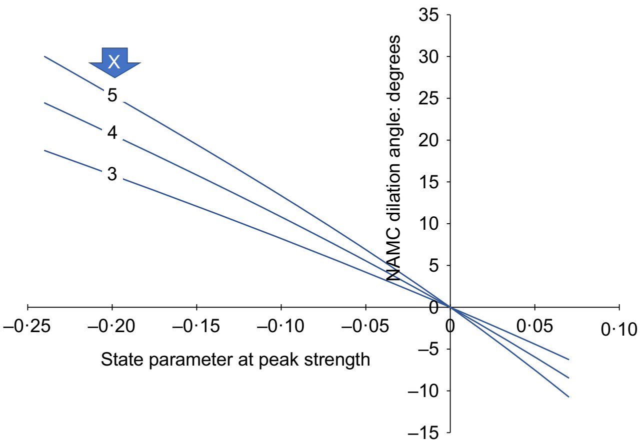

Fig. 8. NAMC dilation angle related to state parameter

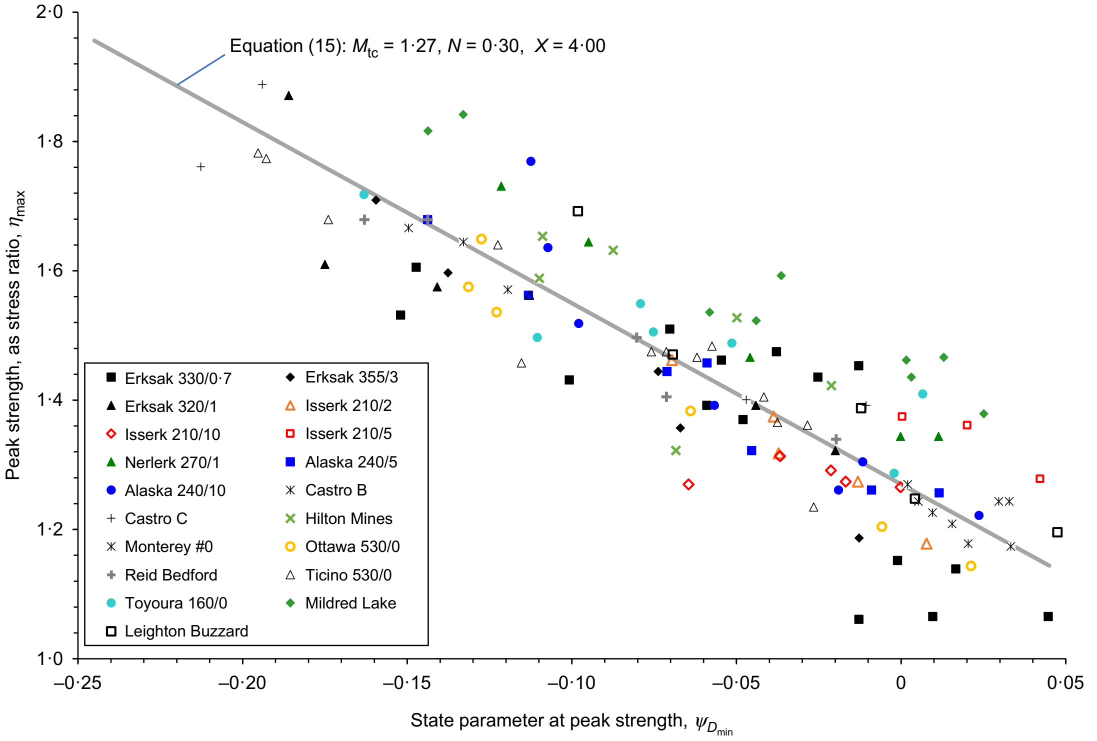

Fig. 9. Strength of sands from Been & Jefferies (1985, 1986) plotted in terms of state parameter at peak strength as opposed to initial state. Trend line drawn for 'not unusual' sand properties, see Table 1 for some individual properties

subsequently illustrated in Jefferies & Been (2006; Fig. 2.13), which can be seen in Fig. 9 with results for each soil scattering around that soil's basic trend given by equation (15) with soil-specific properties. Notwithstanding the effect of fabric, a trendline through the band of $\phi - \psi$ data reported by Been & Jefferies (1985, 1986) can be expressed as $\phi = \phi_c + \alpha\psi$, which, by comparison to equation (15), shows that their proposition amounts to parameter group $(1 - N)X$ being sufficiently near constant as to be much less important than ψ_0. Their study was carried out in the context of hydraulic fills, and while a range of sands for such fills was investigated, in the wider context of 'soil' the proposition is overstated. This now shows up in the application of the state parameter approach to silts where the $(1 - N)X$ can have half the value for uniform sands or, in the case of well-graded sandy silts, where it can have double the value: Table 1. Been & Jefferies (1985, 1986) presented a reasonable engineering approximation for its context, but wider application of CST should be based on equation (15) and with awareness that fabric can influence Ω.

CONCLUSION

The control of limiting dilatancy (and thus soil strength) by the state parameter is a formal mathematical consequence of Casagrande's canonical characterisation linking void ratio to soil constitutive behaviour. Anchored in kinematics, this formal consequence is independent of soil type, being applicable across the spectrum from clays to sands, provided that there are no bonds between the soil particles. The framework further extends from 'strength' to general 'stress–strain' response when $\eta \neq \eta_{max}$ and provides a consistent expression of particulate material behaviour; state–dilatancy (which is controlled by ψ) sets a limit to the evolution of stress–dilatancy (which is controlled by the dissipation of plastic work). Three soil properties are involved in addition to those characterising the CSL: M_{tc}, N, X.

APPENDIX. OPERATING ZERO-DILATANCY FRICTION RATIO M_i CONSTRAINT

The Jefferies & Shuttle (2011) derivation of M_i is in 'state space' and, while sufficient for understanding how stress–dilatancy may evolve, is slightly inconsistent with use of M_i in plasticity models. The issue is that if M_i is used to scale yield surfaces as well as being part of stress–dilatancy, then M_i should be invariant under neutral loading. As neutral loading involves p changing, then so will ψ; if $M_i(\psi)$ a canon of plasticity theory is violated – that yield surfaces cannot evolve in the absence of plastic strain increment. The violation of this canon is avoided by working with the state parameter at the image condition (which is at $D^p = 0$ and values at this condition denoted by the subscript i), which has a single value for any yield surface: thus, the core function is of the form $M_i(\psi_i)$. It is a little pedantic, as ψ does not change greatly as p varies across a yield surface during neutral loading, but it is straightforward to be consistent using a slight modification of Jefferies & Shuttle (2011), as given here. Options have been introduced for loose soils, also described below.

Image state and limiting dilatancy

For a semi-log CSL (and a semi-log local approximation if using other CSL), the state parameter at the image condition is

$$\psi_i = e - e_i = \psi + \lambda \ln(p/p_i) \tag{25}$$

The limiting dilatancy is the same D_{min} whether viewed from ψ or ψ_i, so

$$X\,\psi_{D_{min}} = X_i\,\psi_i \tag{26}$$

Eliminating ψ_i from equation (26) using equation (25), and noting no loss of generality by considering triaxial conditions as p is independent of θ, it is found that

$$X_i = X/(1 - \lambda X/M_{itc}) \tag{27}$$

Notice the implicit nature of equation (27), as X_i depends on M_{itc}, which in turn depends on X_i. The bisection algorithm is used to solve for the initial value of X_i with subsequent steps using the prior value of M_i to update X_i as M_i evolves slowly. But now M_i is needed.

Operating $M_i(\psi_i)$

For dense soils with $\psi_i < 0$, the derivation is as per Jefferies & Shuttle with the substitution of equation (26), which gives

$$M_i = M\,(1 + NX_i\psi_i/M_{tc})\;\forall\psi_i \leq 0 \tag{28}$$

where M is given by equation (16).

For loose soils with $\psi_i > 0$, there are alternative views that depend on the meaning assigned to D_{min}. On one hand, D_{min} can be viewed as the 'maximum dilatancy', which is $D = 0$ for a contractive soil and only develops when the soil reaches its critical state. This is the Bishop–Taylor idea for loose soil, giving

$$\text{Option 0}: M_i = M\;\forall\,\psi_i > 0 \tag{29}$$

On the other hand, D_{min} can be viewed as the current (transient) limit applied to the hardening law; this limit can arise before the soil reaches its critical state and, in particular, influences how static liquefaction develops with loose soil. This then leads to the idea that equation (28) should be symmetric, in essence mapping Fig. 11 of Been & Jefferies (1985), and thus for loose soils

$$\text{Option 1}: M_i = M\,(1 - NX_i\,\psi_i/M_{tc})\;\forall\,\psi_i > 0 \tag{30}$$

Numerical implementations of M_i offer users the choice between the two options as an input 'flag'. All of equations (28)–(30) meet the requirement of no change under neutral loading.

Comments

Options 0 or 1 only arise for loose soil; for dense soil equation (28) applies and wraps Nova's coefficient into the operating stress–dilatancy throughout. The choice between the two options for loose soil is made from calibration to test data: use what best fits the soil tested. Experience is that some soils are better fit with option 0 while others by option 1.

The choice between these options is also affected by how the test data are processed. While dense soil reaches D_{min} at small strain in a triaxial test, the limiting dilation arises at much larger strains with loose soil – and few loose samples deform as right-cylinders even with lubricated platens. Thus, the laboratory data viewed as 'truth' actually involve soil in non-uniform stress conditions: loose soil departs from conditions of 'an element'. There is also a surprising difference in the reported stress–strain behaviour of the test between alternative area corrections applied to convert the measured load–displacement into stress–strain. Thus, choose what seems best, but there is no point being overly dogmatic over the preference for 0 as opposed to 1, as there are uncertainties in appropriate processing of the basic test data. Also, there is a rarely acknowledged effect of void ratio: $M_{tc}(e)$.

NOTATION

Subscripts

- c critical state
- e referenced to void ratio
- NA non-associated (in context of dilation angle)
- tc triaxial compression condition ($\theta = \pi/6$)
- 0 initial condition
- 1, 2, 3 principal directions of stress or strain

Superscripts

- e elastic
- p plastic

Stress variables (bar over or ′ denotes effective)

- p' mean effective stress ($= \bar{\sigma}_m$) (FL^{-2})
- q triaxial deviator stress. $q = \sigma_1 - \sigma_3$ ($= \bar{\sigma}_q$) (FL^{-2})
- η dimensionless distortional stress measure $\eta = \bar{\sigma}_q/\bar{\sigma}_m$ (–)
- θ Lode angle, $\sin(3\theta) = -13{\cdot}5\,\bar{\sigma}_1\,\bar{\sigma}_2\,\bar{\sigma}_3/\bar{\sigma}_q^3$ (rad)
- $\sigma_{1,\,2,\,3}$ principal stresses (FL^{-2})
- $\bar{\sigma}_m$ mean effective stress $\bar{\sigma}_m = (\bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3)/3$ (FL^{-2})
- $\bar{\sigma}_q$ deviatoric stress invariant $\bar{\sigma}_q = [{}^1\!/\!_2(\sigma_1 - \sigma_2)^2 + {}^1\!/\!_2(\sigma_2 - \sigma_3)^2 + {}^1\!/\!_2(\sigma_3 - \sigma_1)^2]^{1/2}$ (FL^{-2})

Strain variables

- D dilatancy, as ratio of strain rates $d\varepsilon_v/d\varepsilon_q$
- D^p plastic dilatancy, as ratio of strain rates $d\varepsilon_v^p/d\varepsilon_q^p$
- $\varepsilon_{1,\,2,\,3}$ principal strains (assumed coaxial with principal stresses)
- ε_q distortional strain measure work conjugate with $\bar{\sigma}_q$, $\varepsilon_q = 1/3\left[(\sin\theta + \sqrt{3}\,\cos\theta)\varepsilon_1 + 2\sin\theta\varepsilon_2 + (\sin\theta - \sqrt{3}\,\cos\theta)\varepsilon_3\right]$
- ε_v volumetric strain $\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

State variables

- e void ratio
- ψ state parameter, $\psi = e - e_c$

Critical state and soil properties (by convention, soil properties in CST are Greek letters)

- M critical friction ratio, equals η_c at the critical state. Varies with Lode angle, value at triaxial compression (M_{tc}) taken as reference soil property
- N Nova's volumetric coupling coefficient in stress–dilatancy
- X scaling factor (soil property) for state dilatancy
- Γ reference void ratio on CSL, defined at $p' = 1$ kPa
- λ slope of CSL in e–$\ln(\sigma_m)$ space for semi-log idealisation
- λ_{10} slope of CSL, but defined on base 10 logarithms ($= 2{\cdot}3\lambda$)

Thermodynamic

- Ω normalised rate of dissipation of plastic work into heat

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